MTH 203: Groups and Symmetry

Homework IV

(Due 07/09)

Problems for submission

1. Establish the assertion in 3.1(iv) of the Lesson Plan.
2. Establish the assertions in the second sentence of 2.3 (iv)(a) and the first sentence of 2.3 (iv)(b) of the Lesson Plan.
3. Let G be a group. Show that if $(ab)^2 = a^2b^2$ for every $a, b \in G$, then G is a abelian.
4. Let G be a group and $H < G$. Then show that the following sets form subgroups of G.
   (a) The set $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.
   (b) The set $Z(G) = \{g \in G \mid gx = xg, \forall x \in G\}$ called the *center of* G.
   (c) The set $C_G(H) = \{g \in G \mid gh = hg, \forall h \in H\}$ called the *centralizer of* H *in* G.
   (d) The set $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ called the *normalizer of* H *in* G.
5. Let H be a group, and $H, K < G$. Show that $H \lhd K$ and $K \lhd G$ does not necessarily imply that $H \lhd G$ by picking a counterexample in D_{2n}, for some suitable $n \geq 3$.

Problems for practice

1. Let $G = D_{2n}$, for $n \geq 3$. Let $H = \langle r^k \rangle$, for $1 \leq k \leq n-1$, and let $K = \langle s \rangle$, where s is any reflection.
   (a) Is $H, K \lhd G$? Explain why, or why not.
   (b) Compute $Z(G)$.
   (c) Compute $N_G(H)$, $N_G(K)$, $C_G(H)$, and $C_G(K)$.
2. Let G be a group, $H < G$, and $N \lhd G$. Then show that:
   (a) $NH < G$. (NH is called the *internal direct product* of N and H.)
   (b) $H \cap N \lhd H$.
   (c) $N \lhd HN$.
   (d) If $H \lhd G$, then $NH \lhd G$.
   (e) If $o(a)$ is finite for some $a \in G$, then $o(Na) \mid o(a)$.
3. Let G be a group, and $H < G$. Then prove that:
   (a) $Z(G) \lhd G$.
   (b) $N(H) < G$.
   (c) $H \lhd N(H)$.
   (d) $N(H)$ is the largest subgroup in which H is normal.
   (e) $H \lhd G$ if, and only if $N(H) = G$.